1. Algebra

1.1 Let's Not Solve Polynomials

Theorem 1.1.1 — Vieta's Formulas. Let

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0,$$

and let the roots of $f(x)$ be $r_1, r_2, \ldots, r_n$. Then, we have

$$\begin{aligned}
s_1 &= r_1 + r_2 + r_3 + \cdots + r_n &&= -\frac{a_{n-1}}{a_n},\\
s_2 &= r_1r_2 + r_1r_3 + r_1r_4 + \cdots + r_{n-2}r_{n-1} &&= \frac{a_{n-2}}{a_n},\\
s_3 &= r_1r_2r_3 + r_1r_2r_4 + \cdots + r_{n-2}r_{n-1}r_n &&= -\frac{a_{n-3}}{a_n},\\
&\vdots\\
s_n &= r_1r_2r_3\cdots r_n &&= (-1)^n\frac{a_0}{a_n},
\end{aligned}$$

where each s_k is the symmetric sum of the roots of $f(x)$ taken k at a time. We call these equations Vieta's Formulas.

1.1.1 Examples

■ **Example 1.1** The polynomial $P(x) = x^3 + ax^2 + bx + c$ has the property that the mean of its zeros, the product of its zeros, and the sum of its coefficients are all equal. If the y-intercept of the graph of $y = P(x)$ is 2, what is b? ■

■ **Example 1.2** Let a, b, c be the roots of $x^3 - 9x^2 + 11x - 1 = 0$, and let $s = \sqrt{a} + \sqrt{b} + \sqrt{c}$. Find $s^4 - 18s^2 - 8s$. ■

■ **Example 1.3** Suppose that the roots of $x^3 + 3x^2 + 4x - 11 = 0$ are a, b, and c, and that the roots of $x^3 + rx^2 + sx + t = 0$ are $a+b$, $b+c$, and $c+a$. (AIME)

1. Find r
2. Find t.

■

■ **Example 1.4** Three of the roots of $x^4 + ax^2 + bx + c = 0$ are $2, -3$, and 5. Find the value of $a+b+c$ (HMMT) ■

■ **Example 1.5** Let α, β, γ be the roots of $x^3 - 3x^2 + 1$ (Misha Lavrov)

1. Find a polynomial whose roots are $\alpha+3, \beta+3$, and $\gamma+3$
2. Find a polynomial whose roots are $\frac{1}{\alpha+3}, \frac{1}{\beta+3}$, and $\frac{1}{\gamma+3}$
3. Compute $\frac{1}{\alpha+3} + \frac{1}{\beta+3} + \frac{1}{\gamma+3}$
4. Find a polynomial whose roots are α^2, β^2, and γ^2