

Part 1 Algebra (Polynomials and complex roots; Solving (systems of) equations; Optimization)

1. Given $f(z) = z^2 - 19z$, there are complex numbers z with the property that z , $f(z)$, and $f(f(z))$ are the vertices of a right triangle in the complex plane with a right angle at $f(z)$. There are positive integers m and n such that one such value of z is $m + \sqrt{n} + 11i$. Find $m + n$.
2. Let $a > 1$ and $x > 1$ satisfy $\log_a(\log_a(\log_a 2) + \log_a 24 - 128) = 128$ and $\log_a(\log_a x) = 256$. Find the remainder when x is divided by 1000.
3. It is given that the roots of the polynomial $P(z) = z^{2019} - 1$ can be written in the form $z_k = x_k + iy_k$ for $1 \leq k \leq 2019$. Let Q denote the monic polynomial with roots equal to $2x_k + iy_k$ for $1 \leq k \leq 2019$. Compute $Q(-2)$.

Part 2 Combinatorics (Expected value; Events with States; Recursion)

4. Let $SP_1P_2P_3EP_4P_5$ be a heptagon. A frog starts jumping at vertex S . From any vertex of the heptagon except E , the frog may jump to either of the two adjacent vertices. When it reaches vertex E , the frog stops and stays there. Find the number of distinct sequences of jumps of no more than 12 jumps that end at E .
5. Misha rolls a standard, fair six-sided die until she rolls 1-2-3 in that order on three consecutive rolls. The probability that she will roll the die an odd number of times is $\frac{m}{n}$ where m and n are relatively prime positive integers. Find $m + n$.
6. There are 100 lightbulbs B_1, \dots, B_{100} spaced evenly around a circle in this order. Additionally, there are 100 switches S_1, \dots, S_{100} such that for all $1 \leq i \leq 100$, switch S_i toggles the states of lights B_{i-1} and B_{i+1} (where here $B_{101} = B_1$). Suppose David chooses whether to flick each switch with probability $\frac{1}{2}$. What is the expected number of lightbulbs which are on at the end of this process given that not all lightbulbs are off?

Part 3 Geometry (Rotation/reflection/homothety; Laws of Sines and Cosines; Cyclic Quads and Ptolemy)

7. Let \overline{AB} be a chord of a circle ω , and let P be a point on the chord \overline{AB} . Circle ω_1 passes through A and P and is internally tangent to ω . Circle ω_2 passes through B and P and is internally tangent to ω . Circles ω_1 and ω_2 intersect at points P and Q . Line PQ intersects ω at X and Y . Assume that $AP = 5$, $PB = 3$, $XY = 11$, and $PQ^2 = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

8. Let $\triangle ABC$ have side lengths $AB = 30$, $BC = 32$, and $AC = 34$. Point X lies in the interior of \overline{BC} , and points I_1 and I_2 are the incenters of $\triangle ABX$ and $\triangle ACX$, respectively. Find the minimum possible area of $\triangle AI_1I_2$ as X varies along \overline{BC} .
9. The area of the smallest equilateral triangle with one vertex on each of the sides of the right triangle with side lengths $2\sqrt{3}$, 5 , and $\sqrt{37}$, as shown, is $\frac{m\sqrt{p}}{n}$, where m , n , and p are positive integers, m and n are relatively prime, and p is not divisible by the square of any prime. Find $m + n + p$.

Part 4 Number Theory (Fermat Little, Euler's and Wilson theorems; Order of a number modulo p)

10. Find the least positive integer n such that when 3^n is written in base 143, its two right-most digits in base 143 are 01.
11. Find the least positive integer m such that $m^2 - m + 11$ is a product of at least four not necessarily distinct primes.